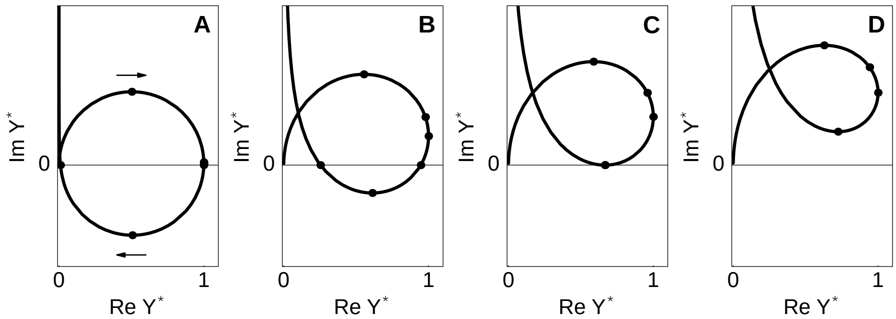
Handbook
of
Electrochemical Impedance Spectroscopy



CIRCUITS
made of
RESISTORS, INDUCTORS and
CAPACITORS

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Contents

Chapter 1

Circuits made of R, L and C

1.1 (L+(R/C)) circuit

1.1.1 Circuit

Fig. 1.1.

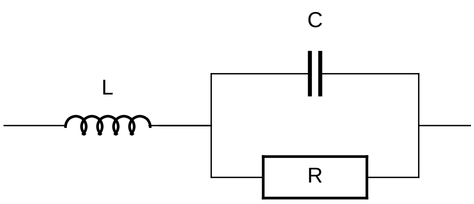

Figure 1.1: Circuit (L+(R/C)).

1.1.2 Impedance

$$Z(\omega) = L\,\mathrm{i}\,\omega + \frac{R}{1 + R\,C\,\mathrm{i}\,\omega}$$

$$\mathrm{Re}\;Z(\omega) = \frac{R}{C^2R^2\omega^2 + 1},\;\mathrm{Im}\;Z(\omega) = \omega\left(L - \frac{CR^2}{C^2R^2\omega^2 + 1}\right)$$

1.1.3 Reduced impedance

$$Z^*(u) = \frac{Z(u)}{R} = \mathrm{i}\,T\,u + \frac{1}{1 + \mathrm{i}\,u},\; u = R\,C\,\omega,\; T = \frac{L}{C\,R^2} \tag{1.1}$$

$$\mathrm{Re}\;Z^*(u) = \frac{1}{u^2 + 1},\;\mathrm{Im}\;Z^*(u) = u\left(T - \frac{1}{u^2 + 1}\right)$$

Reduced characteristic angular frequency $u_\mathrm{c} = 1$ with:

$$\mathrm{Re}\;Z(u_\mathrm{c}) = 1/2,\;\mathrm{Im}\;Z(u_\mathrm{c}) = T - 1/2$$

$T < 1 \Rightarrow$:

- $u_{\mathrm{Im}\ Z=0} = \sqrt{\frac{1-T}{T}}$, $\mathrm{Re}\ Z(u_{\mathrm{Im}\ Z=0}) = T$.
- reduced angular frequency at the apex :

$$u_{\mathrm{a}} = \frac{1}{\sqrt{2}} \sqrt{\frac{-2T + \sqrt{8T+1} - 1}{T}}$$

with:

$$\mathrm{Re}\ Z(u_{\mathrm{a}}) = \frac{1}{4}\left(\sqrt{8T+1}+1\right)$$

$$\mathrm{Im}\ Z(u_{\mathrm{a}}) = \frac{\sqrt{T}\left(\sqrt{8T+1}-3\right)\sqrt{-2T+\sqrt{8T+1}-1}}{\sqrt{2}\left(\sqrt{8T+1}-1\right)}$$

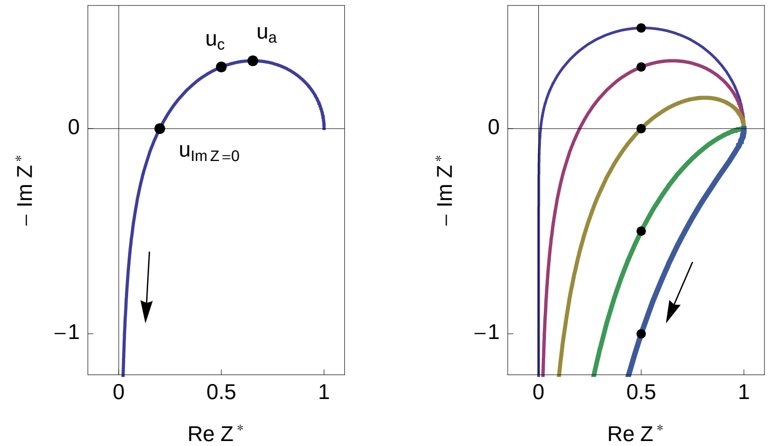

Figure 1.2: Nyquist diagram of the reduced impedance for the (L+(R/C)) circuit (Fig. 1.1, Eq. (1.2)) plotted for $T = 0.2$ (left) and $T = 0.01, 0.2, 0.5, 1, 1.5$ (right). The line thickness increases with increasing T. Dots: reduced characteristic angular frequency $u_c = 1$ (right).

1.2 (R_0+(L+(R/C))) circuit

Fig. 1.3.

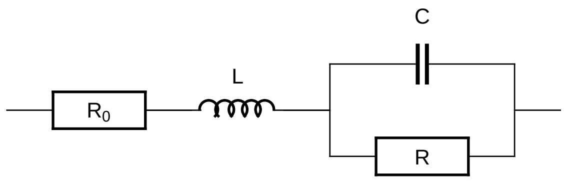

Figure 1.3: Circuit (R_0+(L+(R/C))).

1.2.1 Impedance

$$Z(\omega) = R_0 + L\,\mathrm{i}\,\omega + \frac{R}{1 + R\,C\,\mathrm{i}\,\omega}$$

$$\mathrm{Re}\; Z(\omega) = R_0 + \frac{R}{C^2 R^2 \omega^2 + 1},\ \mathrm{Im}\; Z(\omega) = \omega \left(L - \frac{C R^2}{C^2 R^2 \omega^2 + 1} \right)$$

1.2.2 Reduced impedance

$$Z^*(u) = \frac{Z(u)}{R} = \rho + \mathrm{i}\,T\,u + \frac{1}{1 + \mathrm{i}\,u},\ u = R\,C\,\omega,\ \rho = \frac{R_0}{R},\ T = \frac{L}{C\,R^2} \quad (1.2)$$

$$\mathrm{Re}\; Z^*(u) = \rho + \frac{1}{u^2 + 1},\ \mathrm{Im}\; Z^*(u) = u \left(T - \frac{1}{u^2 + 1} \right)$$

Reduced characteristic angular frequency $u_{\mathrm{c}} = 1$ with:

$$\mathrm{Re}\; Z(u_{\mathrm{c}}) = \rho + 1/2,\ \mathrm{Im}\; Z(u_{\mathrm{c}}) = T - 1/2$$

$T < 1 \Rightarrow$:

- $u_{\mathrm{Im}\; Z=0} = \sqrt{\frac{1-T}{T}}$, $\mathrm{Re}\; Z(u_{\mathrm{Im}\; Z=0}) = \rho + T$.
- reduced angular frequency at the apex :

$$u_{\mathrm{a}} = \frac{1}{\sqrt{2}} \sqrt{\frac{-2T + \sqrt{8T+1} - 1}{T}}$$

with:

$$\mathrm{Re}\; Z(u_{\mathrm{a}}) = \rho + \frac{1}{4} \left(\sqrt{8T+1} + 1 \right)$$

$$\mathrm{Im}\; Z(u_{\mathrm{a}}) = \frac{\sqrt{T} \left(\sqrt{8T+1} - 3 \right) \sqrt{-2T + \sqrt{8T+1} - 1}}{\sqrt{2} \left(\sqrt{8T+1} - 1 \right)}$$

1.3 (C+(R/L)) circuit

1.3.1 Circuit

Fig. 1.5.

1.3.2 Impedance

$$Z(\omega) = \frac{1}{C\,\mathrm{i}\,\omega} + \frac{L\,R\,\mathrm{i}\,\omega}{R + L\,\mathrm{i}\,\omega}$$

$$\mathrm{Re}\; Z(\omega) = \frac{L^2\,R\,\omega^2}{L^2\,\omega^2 + R^2},\ \mathrm{Im}\; Z(\omega) = \frac{L\,R^2\,\omega}{L^2\,\omega^2 + R^2} - \frac{1}{C\,\omega}$$

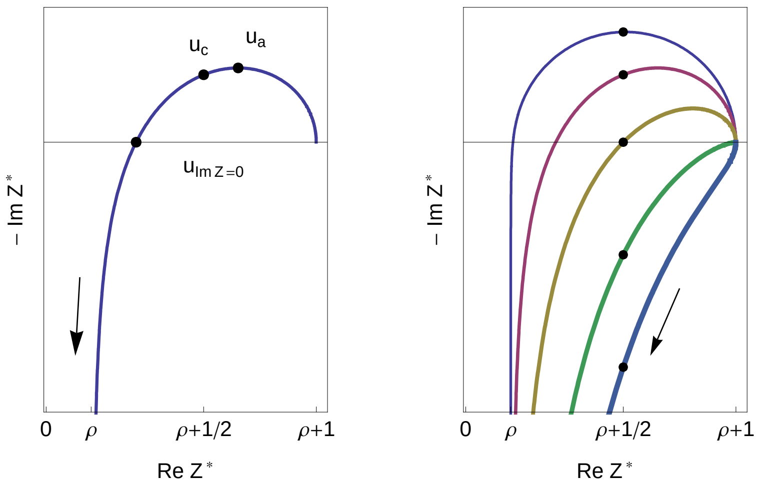

Figure 1.4: Nyquist diagram of the reduced impedance for the (R_0+(L+(R/C))) circuit (Fig. 1.1, Eq. (1.2)) plotted $\rho = 0.2$ and $T = 0.2$ (left) and $T = 0.01, 0.2, 0.5, 1, 1.5$ (right). The line thickness increases with increasing T. Dots: reduced characteristic angular frequency $u_c = 1$ (right).

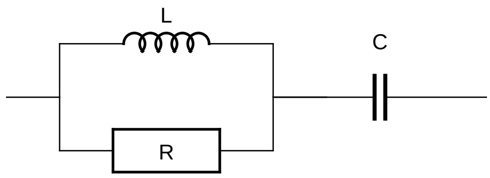

Figure 1.5: Circuit (C+(R/L)).

1.3.3 Reduced impedance

$$Z^*(u) = \frac{Z(u)}{R} = \frac{1}{\mathrm{i}\,T\,u} + \frac{\mathrm{i}\,u}{1+\mathrm{i}\,u},\; u = \frac{L}{R}\,\omega,\; T = \frac{R^2\,C}{L} \tag{1.3}$$

$$\mathrm{Re}\;Z^*(u) = \frac{u^2}{u^2+1},\;\mathrm{Im}\;Z^*(u) = \frac{u}{u^2+1} - \frac{1}{Tu}$$

Reduced characteristic angular frequency $u_c = 1$ with:

$$\mathrm{Re}\;Z(u_c) = 1/2,\;\mathrm{Im}\;Z(u_c) = 1/2 - 1/T$$

$T > 1 \Rightarrow$:

- $u_{\mathrm{Im}\;Z=0} = \dfrac{1}{\sqrt{T-1}},\;\mathrm{Re}\;Z(u_{\mathrm{Im}\;Z=0}) = \dfrac{1}{T}.$
- reduced angular frequency at the apex :

$$u_a = \frac{1}{\sqrt{2}}\sqrt{\frac{T+\sqrt{T+8}\sqrt{T}+2}{T-1}}$$

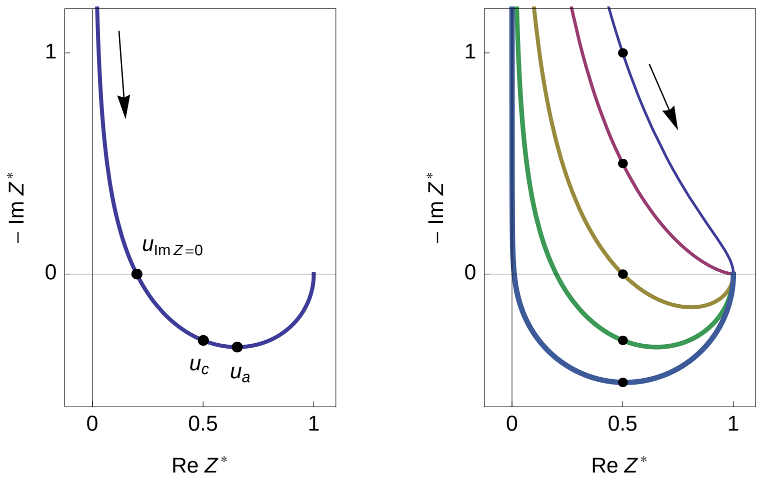

Figure 1.6: Nyquist diagram of the reduced impedance for the (C+(R/L)) circuit (Fig. 1.5, Eq. (1.3)) plotted for $T = 5$ (left) and $T = 0.66, 1, 2, 5, 100$ (right). The line thickness increases with increasing T. Dots: reduced characteristic angular frequency $u_c = 1$ (right).

with:

$$\mathrm{Re}\, Z(u_a) = \frac{1}{4}\left(\frac{1}{\sqrt{\frac{T}{T+8}}} + 1\right)$$

$$\mathrm{Im}\, Z(u_a) = \frac{\sqrt{2}(T-1)\left(\sqrt{T} + \sqrt{T+8}\right)}{T\left(3\sqrt{T} + \sqrt{T+8}\right)\sqrt{\frac{T+\sqrt{T+8}\sqrt{T}+2}{T-1}}}$$

1.4 (R/L)+(R/C) circuit

Fig. 1.7.

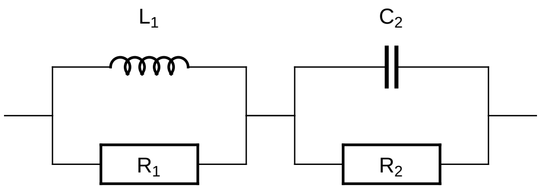

Figure 1.7: Circuit (R/L)+(R/C).

1.4.1 Impedance

$$Z(\omega) = \frac{L_1 R_1 \mathrm{i}\,\omega}{L_1 \mathrm{i}\,\omega + R_1} + \frac{R_2}{C_2 R_2 \mathrm{i}\,\omega + 1} = \frac{R_1\, \tau_1\, \mathrm{i}\,\omega}{1 + \tau_1\, \mathrm{i}\,\omega} + \frac{R_2}{1 + \tau_2\, \mathrm{i}\,\omega},\ \tau_1 = \frac{L_1}{R_1},\ \tau_2 = R_2\, C_2 \tag{1.4}$$

$$\mathrm{Re}\, Z(\omega) = R_1 \left(1 - \frac{1}{\tau_1^2\omega^2+1}\right) + \frac{R_2}{\tau_2^2\omega^2+1}, \;\; \mathrm{Im}\, Z(\omega) = \frac{R_1\tau_1\omega}{\tau_1^2\omega^2+1} - \frac{R_2\tau_2\omega}{\tau_2^2\omega^2+1}$$

$$\lim_{\omega\to 0} Z(\omega) = R_2, \;\; \lim_{\omega\to\infty} Z(\omega) = R_1$$

1.4.2 Reduced impedance

$$Z^*(u) = \frac{Z(\omega)}{R_2} = \rho\,\frac{\mathrm{i}\,u}{1+\mathrm{i}\,u} + \frac{1}{1+T\,\mathrm{i}\,u}, \; u = \omega\,\tau_1, \; \rho = \frac{R_1}{R_2}, \; T = \frac{\tau_2}{\tau_1}$$

$$\mathrm{Re}\, Z^*(u) = \frac{u^2\rho}{u^2+1} + \frac{1}{T^2u^2+1}, \;\; \mathrm{Im}\, Z^*(u) = \frac{u\rho}{u^2+1} - \frac{Tu}{T^2u^2+1}$$

1.4.3 Nyquist impedance diagrams

- $T > 1$, Fig. 1.8.

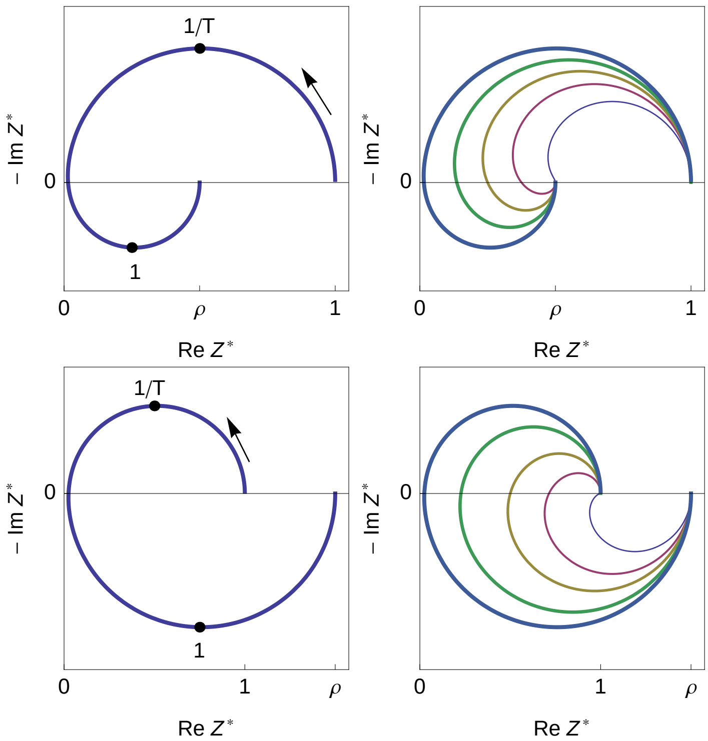

Figure 1.8: $T > 1$. Nyquist diagrams of the impedance for the (R/L)+(R/C) circuit (Fig. 1.7, Eq. (1.4)) plotted for : top : $\rho < 1$ ($\rho = 0.5$), bottom : $\rho > 1$ ($\rho = 1.5$). $T \gg 1$ ($T = 10^2$) (left) and increasing values of T (right). The line thickness increases with increasing T.

- $T < 1$, Fig. 1.9.
- $T = 1$, Fig. 1.10.

$$T = 1 \Rightarrow Z^*(u) = \frac{1+\rho\,\mathrm{i}\,u}{1+\mathrm{i}\,u}$$

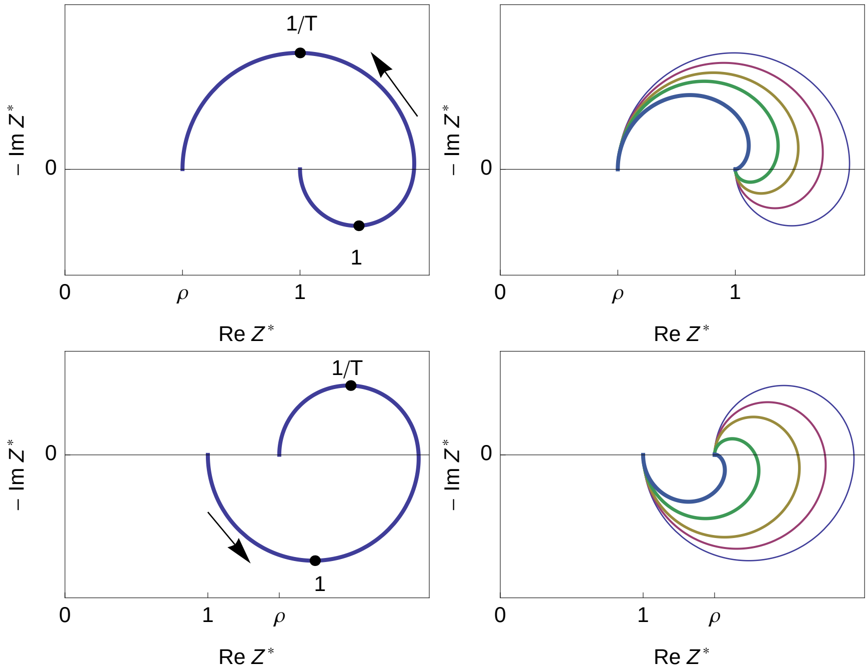

Figure 1.9: $T < 1$. Nyquist diagrams of the impedance for the (R/L)+(R/C) circuit (Fig. 1.7, Eq. (1.4)) plotted for : top : $\rho < 1$ ($\rho = 0.5$), bottom : $\rho > 1$ ($\rho = 1.5$). $T \ll 1$ ($T = 10^{-2}$) (left) and increasing values of T (right). The line thickness increases with increasing T.

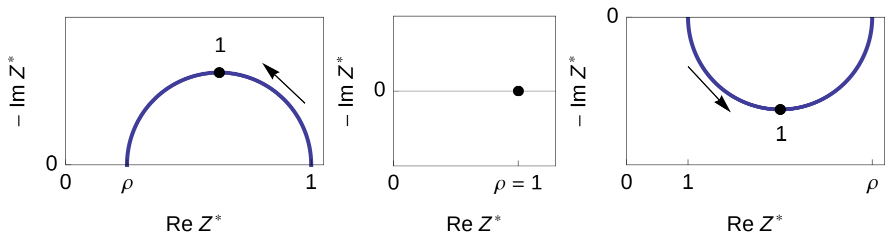

Figure 1.10: $T = 1$. Nyquist diagrams of the impedance for the (R/L)+(R/C) circuit. Left: $\rho < 1$, middle: $\rho = 1$, right: $\rho > 1$.

1.4.4 Inductive and capacitive Nyquist diagrams

$$T > 1 \text{ and } \frac{1}{T} < \rho < T \text{ or } T < 1 \text{ and } T < \rho < \frac{1}{T}$$
$$\Rightarrow u_{\text{Im}=0} = \frac{\sqrt{T-\rho}}{\sqrt{T(T\rho-1)}},\ \text{Re}_{\text{Im}=0} = \frac{1+\rho}{1+T}$$

Fig. 1.11.

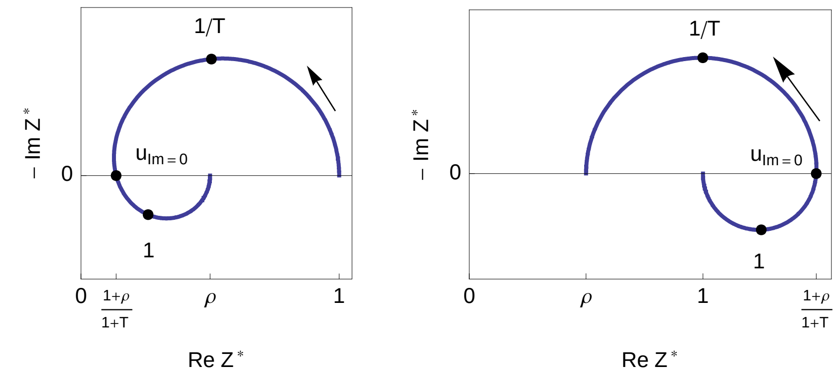

Figure 1.11: Inductive and capacitive Nyquist diagrams. Left : $T > 1$ and $\frac{1}{T} < \rho < T$, right : $T < 1$ and $T < \rho < \frac{1}{T}$.

1.4.5 $\rho = R_1/R_2 = 1$

$$Z^*(u) = \frac{\mathrm{i}\,u}{1 + \mathrm{i}\,u} + \frac{1}{1 + T\,\mathrm{i}\,u}$$

Fig. 1.12.

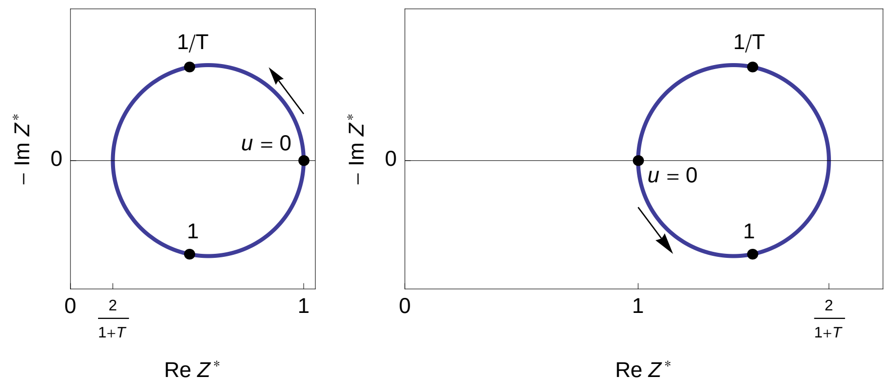

Figure 1.12: $\rho = R_1/R_2 = 1$. Nyquist diagram: full circle. Left: $T > 1$, right $T < 1$.

1.4.6 Array of Nyquist impedance diagrams

Fig. 1.13.

1.5 RLC parallel circuit

1.5.1 Circuit

Fig. 1.14.

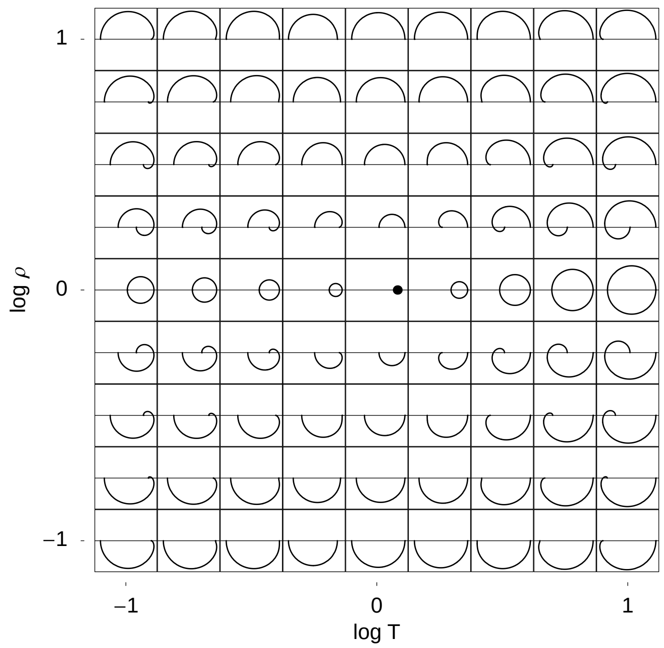

Figure 1.13: Array of Nyquist impedance diagrams for the (R/L)+(R/C) circuit. $T = \rho = 1 \Rightarrow Z^*(u) = 1,\ \forall u$.

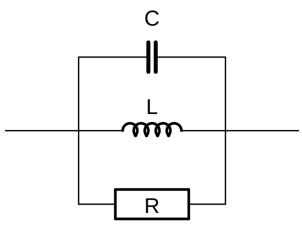

Figure 1.14: Circuit ((R/L)/C).

1.5.2 Admittance

$$Y(\omega) = \frac{1}{L\,\mathrm{i}\,\omega} + C\,\mathrm{i}\,\omega + \frac{1}{R} = \frac{L\,\mathrm{i}\,\omega + R + C\,L\,R\,(\mathrm{i}\,\omega)^2}{L\,R\,\mathrm{i}\,\omega}$$
$$\mathrm{Re}\,Y(\omega) = \frac{1}{R},\ \mathrm{Im}\,Y(\omega) = -\frac{1}{L\,\omega} + C\,\omega$$

$\mathrm{Re}\,Y(\omega)$ is constant, $\lim_{\omega\to 0} \mathrm{Im}\,Y(\omega) = -\infty$, $\lim_{\omega\to\infty} \mathrm{Im}\,Y(\omega) = \infty \Rightarrow$ Nyquist diagram of $Y(\omega)$ is a vertical straight line.

1.5.3 Reduced admittance

$$Y^*(u) = R\,Y(u) = 1 + \Lambda\left(\mathrm{i}\,u + \frac{1}{\mathrm{i}\,u}\right),\ u = \omega\sqrt{L\,C},\ \Lambda = R\sqrt{\frac{C}{L}}$$

$$\mathrm{Re}\ Y^*(u) = 1,\ \mathrm{Im}\ Y^*(u) = \Lambda\left(u - \frac{1}{u}\right)$$

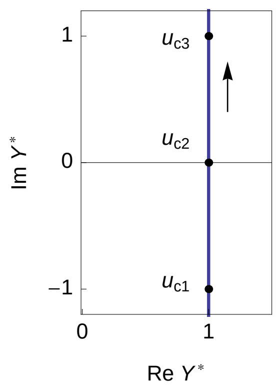

Figure 1.15: Nyquist diagram of the ((R/L)/C) circuit reduced admittance. $u_{c1} = (-1+\sqrt{1+4\,\Lambda^2})/2\,\Lambda,\ u_{c2} = 1,\ u_{c3} = (1+\sqrt{1+4\,\Lambda^2})/2\,\Lambda$.

1.5.4 Impedance

$$Z(\omega) = \frac{1}{Y(\omega)} = \frac{1}{\dfrac{1}{L\,\mathrm{i}\,\omega} + C\,\mathrm{i}\,\omega + \dfrac{1}{R}} = \frac{L\,R\,\mathrm{i}\,\omega}{L\,\mathrm{i}\,\omega + R + C\,L\,R\,(\mathrm{i}\,\omega\,)^2}$$

$$\mathrm{Re}\ Z(\omega) = \frac{L^2\,R\,\omega^2}{L^2\,\omega^2 + \left(R - C\,L\,R\,\omega^2\right)^2},\ \mathrm{Im}\ Z(\omega) = \frac{L\,R^2\,\omega\,\left(1 - C\,L\,\omega^2\right)}{L^2\,\omega^2 + R^2\left(-1 + C\,L\,\omega^2\right)^2}$$

The Nyquist diagram of $Y(\omega)$ is a vertical straight line $\Rightarrow$ the Nyquist diagram of $Z(\omega)$ is a full circle.

1.5.5 Reduced impedance

$$Z^*(u) = \frac{Z(u)}{R} = \frac{\mathrm{i}\,u}{\Lambda + \mathrm{i}\,u + \Lambda\,(\mathrm{i}\,u)^2},\ u = \omega\sqrt{L\,C},\ \Lambda = R\sqrt{\frac{C}{L}}$$

$$\mathrm{Re}\ Z^*(u) = \frac{u^2}{u^2 + \Lambda^2\,(1-u^2)^2},\ \mathrm{Im}\ Z^*(u) = \frac{\Lambda\,u(1-u^2)}{u^2 + \Lambda^2(1-u^2)^2}$$

1.6 RLC serie circuit

1.6.1 Circuit

Fig. 1.17.

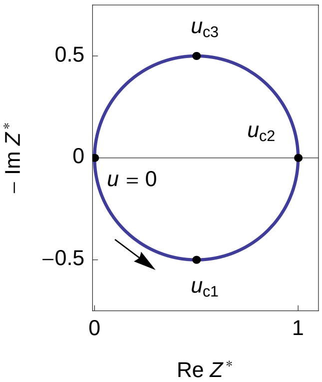

Figure 1.16: Nyquist diagram of the ((R/L)/C) circuit reduced impedance. $u_{c1} = (-1+\sqrt{1+4\Lambda^2})/2\Lambda,\ u_{c2} = u_r = 1,\ u_{c3} = (1+\sqrt{1+4\Lambda^2})/2\Lambda\ (u_{c3} - u_{c1} = \Lambda)$.

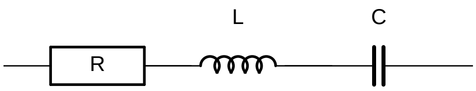

Figure 1.17: Circuit ((R+L)+C).

1.6.2 Impedance

$$Z(\omega) = R + L\,\mathrm{i}\,\omega + \frac{1}{C\,\mathrm{i}\,\omega} = \frac{1 + R\,C\,\mathrm{i}\,\omega + C\,L\,(\mathrm{i}\,\omega)^2}{C\,\mathrm{i}\,\omega}$$
$$\mathrm{Re}\ Z(\omega) = R,\ \mathrm{Im}\ Z(\omega) = -\frac{1}{C\,\omega} + L\,\omega$$

Re $Z(\omega)$ is constant, $\lim_{\omega\to 0} \mathrm{Im}\ Z(\omega) = -\infty$, $\lim_{\omega\to\infty} \mathrm{Im}\ Z(\omega) = \infty \Rightarrow$ Nyquist diagram of $Z(\omega)$ is a vertical straight line.

1.6.3 Reduced impedance

$$Z^*(u) = \frac{Z(u)}{R} = 1 + \frac{1}{\Lambda}\left(\mathrm{i}\,u + \frac{1}{\mathrm{i}\,u}\right),\ u = \omega\sqrt{L\,C},\ \Lambda = R\sqrt{\frac{C}{L}}$$
$$\mathrm{Re}\ Z^*(u) = 1,\ \mathrm{Im}\ Z^*(u) = \frac{1}{\Lambda}\left(u - \frac{1}{u}\right)$$

1.6.4 Admittance

$$Y(\omega) = \frac{1}{Z(\omega)} = \frac{C\,\mathrm{i}\,\omega}{1 + R\,C\,\mathrm{i}\,\omega + C\,L\,(\mathrm{i}\,\omega)^2}$$
$$\mathrm{Re}\ Y(\omega) = -\frac{C^2\,R\,\omega^2}{C^2\,R^2\,\omega^2 + (-1 + C\,L\,\omega^2)^2},\ \mathrm{Im}\ Y(\omega) = \frac{C\,\omega\,\left(1 - C\,L\,\omega^2\right)}{1 + C\,\omega^2\,\left(C\,R^2 + L\,\left(-2 + C\,L\,\omega^2\right)\right)}$$

The Nyquist diagram of $Z(\omega)$ is a vertical straight line $\Rightarrow$ the Nyquist diagram of $Y(\omega)$ is a full circle.

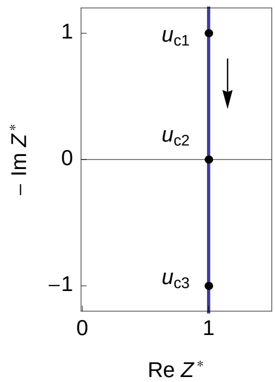

Figure 1.18: Nyquist diagram of the ((R+L)+C) circuit reduced impedance. $u_{c1} = (-\Lambda + \sqrt{4+\Lambda^2})/2,\ u_{c2} = 1,\ u_{c3} = (\Lambda + \sqrt{4+\Lambda^2})/2$.

1.6.5 Reduced admittance

$$Y^*(u) = R\,Y(u) = \frac{\Lambda\,\mathrm{i}\,u}{1+\Lambda\,\mathrm{i}\,u+(\mathrm{i}\,u)^2},\ u = \omega\sqrt{L\,C},\ \Lambda = R\sqrt{\frac{C}{L}}$$

$$\mathrm{Re}\ Y^*(u) = \frac{u^2\,\Lambda^2}{1+u^4+u^2\ (-2+\Lambda^2)},\ \mathrm{Im}\ Y^*(u) = \frac{u\,\Lambda\,(1-u^2)}{1+u^4+u^2\ (-2+\Lambda^2)}$$

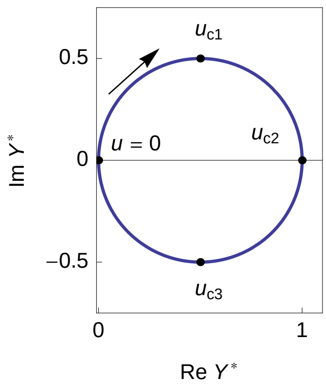

Figure 1.19: Nyquist diagram of the ((R+L)+C) circuit reduced admittance. $u_{c1} = (-\Lambda + \sqrt{4+\Lambda^2})/2,\ u_{c2} = u_r = 1,\ u_{c3} = (\Lambda + \sqrt{4+\Lambda^2})/2,\ (u_{c3} - u_{c1} = \Lambda)$.

1.7 R_0 +RLC parallel circuit

1.7.1 Circuit

Fig. 1.20.

1.7.2 Impedance

$$Z(\omega) = R_0 + \frac{L\,R\,\mathrm{i}\omega}{L\,\mathrm{i}\omega + R + C\,L\,R\,(\mathrm{i}\,\omega)^2} \tag{1.5}$$

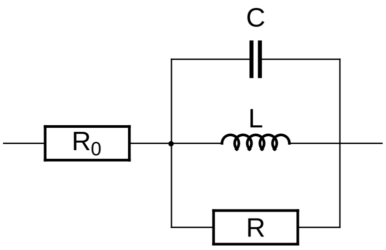

Figure 1.20: R_0+ RLC parallel circuit.

$$Z^*(u) = \frac{Z(u)}{R} = \rho + \frac{\mathrm{i}\,u}{\Lambda + \mathrm{i}\,u + \Lambda\,(\mathrm{i}\,u)^2},\ \rho = \frac{R_0}{R},\ u = \omega\sqrt{L\,C},\ \Lambda = R\sqrt{\frac{C}{L}}$$

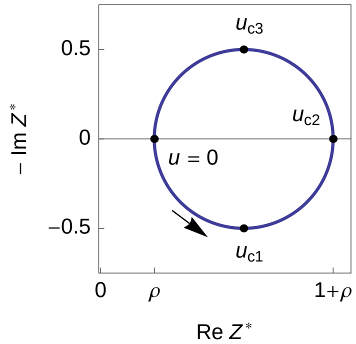

Figure 1.21: Nyquist reduced impedance diagram of the R_0+ RLC parallel circuit. $u_{c1} = (-1+\sqrt{1+4\,\Lambda^2})/2\,\Lambda,\ u_{c2} = u_r = 1,\ u_{c3} = (1+\sqrt{1+4\,\Lambda^2})/2\,\Lambda\ (u_{c3} - u_{c1} = \Lambda)$.

1.8 Transformation formulae (R_1/L_1)+(R_1/C_2) $\to r_1$+ RLC parallel

r_1+ $r_2/l_2/c_2$ parallel circuit is not-distinguishable from (R_1/L_1)+(R_1/C_2) circuit for $R_1^2 C_2/L_2 > 1$ (Fig. 1.22).

$$Z(p) = \frac{R_1\left(C_2 L_1 p^2 + \frac{2L_1 p}{R_1} + 1\right)}{C_2 L_1 p^2 + \frac{p\left(C_2 R_1^2 + L_1\right)}{R_1} + 1} \tag{1.6}$$

$$z(p) = \frac{r_1\left(c_2 l_2 p^2 + \frac{l_2 p (r_1 + r_2)}{r_1 r_2} + 1\right)}{c_2 l_2 p^2 + \frac{l_2 p}{r_2} + 1} \tag{1.7}$$

$$c_2 = \frac{C_2 L_1}{L_1 - C_2 R_1^2},\ l_2 = L_1 - C_2 R_1^2,\ r_2 = R_1\left(\frac{2L_1}{C_2 R_1^2 + L_1} - 1\right) \tag{1.8}$$

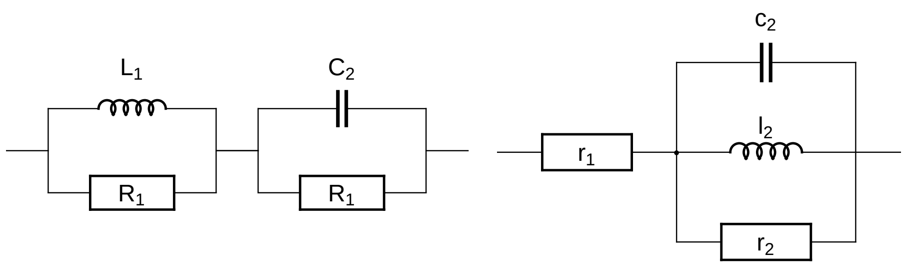

Figure 1.22: $(R_1/L_1)+(R_1/C_2)$ ($(R_1/L_1)+(R_2/C_2)$ circuit with $R_1 = R_2$) and $r_1+r_2/l_2/c_2$ parallel circuit.

Chapter 2

Quartz resonator

2.1 BVD equivalent circuit

Fig. 2.1, [3, 5, 6, 1].

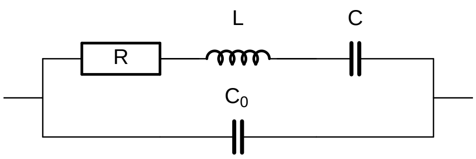

Figure 2.1: BVD (Butterworth-van Dyke)-equivalent circuit of a quartz resonator.

2.2 Admittance

$$Y(\omega) = \frac{1}{R + L\,\mathrm{i}\,\omega + \frac{1}{C\,\mathrm{i}\,\omega}} + \mathrm{i}\,\omega\, C_0 = \mathrm{i}\,\omega \left(\frac{C}{1 + C\,\mathrm{i}\,\omega\; (L\,\mathrm{i}\,\omega + R)} + C_0 \right)$$

$$\mathrm{Re}\, Y(\omega) = \frac{C^2\, R\, \omega^2}{C^2\, R^2\, \omega^2 + \left(-1 + C\, L\, \omega^2\right)^2},\ \mathrm{Im}\, Y(\omega) = \omega \left(\frac{C\,(1 - C\, L\, \omega^2)}{1 + C\,\omega^2\; (C\, R^2 + L\; (-2 + C\, L\, \omega^2))} + C_0 \right)$$

2.3 Reduced admittance

$$Y^*(u) = R\, Y(u) = \frac{\Lambda\,\mathrm{i}\,u}{1 + \Lambda\,\mathrm{i}\,u + (\mathrm{i}\,u)^2} + \gamma\,\mathrm{i}\,u,\ u = \omega\,\sqrt{L\,C},\ \Lambda = R\,\sqrt{\frac{C}{L}},\ \gamma = \frac{R\, C_0}{\sqrt{L\,C}}$$

$$\mathrm{Re}\, Y^*(u) = \frac{u^2\,\Lambda^2}{1 + u^4 + u^2\;(-2 + \Lambda^2)},\ \mathrm{Im}\, Y^*(u) = u\,\gamma + \frac{u\,\Lambda\,(1 - u^2)}{1 + u^4 + u^2\;(-2 + \Lambda^2)}$$

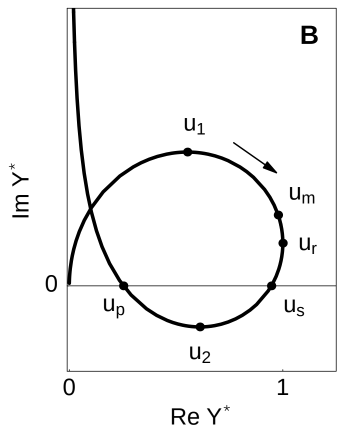

Figure 2.2: Definitions for u_1, u_m, u_r, u_s, u_2 and u_p.

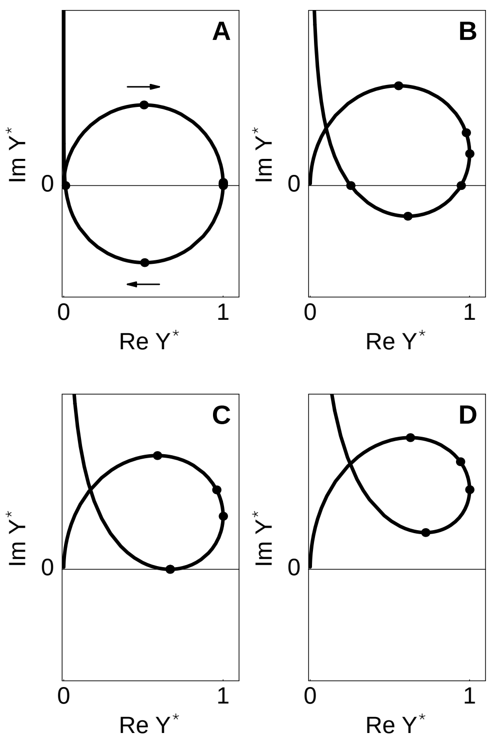

Figure 2.3: Change of admittance diagram with γ. $\Lambda = 1$, $\gamma = 10^{-2}$ (A), 2×10^{-1} (B), 1/3 (C), 1/2 (D).

2.3.1 Characteristic frequencies

- Maximum of the real part of Y^* for:
  $u_\mathrm{r} = 1 \Rightarrow \mathrm{Re}\, Y^*(u_\mathrm{r}) = 1, \mathrm{Im}\, Y^*(u_\mathrm{r}) = \gamma$

- Zero-phase reduced angular frequencies: $u_{\rm s}$ and $u_{\rm p}$ defined for $\gamma < 1/(2+\Lambda)$:

  1. $\gamma < \dfrac{1}{2+\Lambda} \Rightarrow$

$$u_{\rm s} = \sqrt{\frac{\Lambda - \gamma\,(-2+\Lambda^2) - \Lambda\,\sqrt{1-2\,\gamma\,\Lambda + \gamma^2\,(-4+\Lambda^2)}}{2\,\gamma}},$$
$$\mathrm{Re}\,Y^*(u_{\rm s}) = \frac{1}{2}\left(1+\gamma\Lambda + \sqrt{(\gamma(\Lambda-2)-1)(\gamma(\Lambda+2)-1)}\right)$$
$$u_{\rm p} = \sqrt{\frac{2\,\gamma + \Lambda - \gamma\,\Lambda^2 + \Lambda\,\sqrt{1-2\,\gamma\,\Lambda + \gamma^2\,(-4+\Lambda^2)}}{2\,\gamma}},$$
$$\mathrm{Re}\,Y^*(u_{\rm p}) = \frac{1}{2}\left(1+\gamma\Lambda - \sqrt{(\gamma(\Lambda-2)-1)(\gamma(\Lambda+2)-1)}\right)$$

  2. $\gamma = \dfrac{1}{2+\Lambda} \Rightarrow$

$$u_{\rm s} = u_{\rm p} = \frac{-\gamma\Lambda^2 + 2\gamma + \Lambda}{2\gamma}$$
$$\mathrm{Re}\,Y^*(u_{\rm s}) = \mathrm{Re}\,Y^*(u_{\rm p}) = \frac{1}{2}(1+\gamma\Lambda) \quad \text{(Fig. 2.3C)}.$$

  3. $\gamma > \dfrac{1}{2+\Lambda} \Rightarrow$
     no zero-phase reduced angular frequency (Fig. 2.3D).

Real quartz : $C_0 \approx 10^{-12}$ F, $C \approx 10^{-14}$ F, $R \approx 100\ \Omega$, $L \approx \times 10^{-2}$ H $\Rightarrow \Lambda \approx 10^{-4}$ and $\gamma \approx 10^{-2}$ [4, 2].

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