How to fit transmission lines with ZFit

I – INTRODUCTION
ZFit is the impedance fitting tool of EC-Lab®. This note will describe how to fit transmission lines using one equivalent circuit elements contained in ZFit.

It has been well known for a long time that the Warburg impedance is equivalent to that of a semi-infinite large network i.e. a transmission line, as shown in Fig. 1 [1, 2].

![Figure 1: The equivalent circuit of the War-Burg impedance.](image)

More recently it has been shown [3] that the impedance of a L-long transmission line made of \( \chi \) and \( \zeta \) elements and terminated by a \( Z_L \) element (Fig. 2) is given by the general expression:

\[
Z = \frac{(\zeta\chi - Z_L^2) \text{sh} \left( \frac{L\sqrt{\chi}}{\sqrt{\zeta}} \right)}{Z_L \text{sh} \left( \frac{L\sqrt{\chi}}{\sqrt{\zeta}} \right) + \sqrt{\zeta\chi} \text{ch} \left( \frac{L\sqrt{\chi}}{\sqrt{\zeta}} \right)} + Z_L
\]

Hereafter, some transmission lines are described and the corresponding "simple" equivalent circuit elements are shown. Firstly, the open-circuited transmission lines will be explained, followed by short-circuited and semi-infinite transmission lines.

III – OPEN-CIRCUITED TRANSMISSION LINES \( Z_L = 1 \)

II - 1  OPEN-CIRCUITED URC (UNIFORM DISTRIBUTED RC)
Let us consider the open-circuited transmission line made of \( r \) and \( c \) elements (Fig. 3).

![Figure 3: L-long open uniform distributed RC (URC) transmission line [4,5].](image)

With three limiting cases
- open-circuited transmission line

\[
Z_L = \infty \Rightarrow Z = \sqrt{\zeta\chi} \coth \left( \frac{L\sqrt{\chi}}{\sqrt{\zeta}} \right)
\]

(1)

- short-circuited transmission line

\[
Z_L = 0 \Rightarrow Z = \sqrt{\zeta\chi} \tanh \left( \frac{L\sqrt{\chi}}{\sqrt{\zeta}} \right)
\]

(2)

- semi-infinite transmission line

\[
L \rightarrow \infty \Rightarrow Z = \sqrt{\zeta\chi}
\]

(3)

\[1\] The transmission lines are named accordingly to the U-\( \chi \zeta \) format where \( U \) means uniformly distributed and \( \chi \) and \( \zeta \) are the elements of the transmission line.
Using Eq. (1), the transmission line impedance is given by:

\[ Z = \sqrt{r} \frac{\coth(L\sqrt{rc\omega})}{\sqrt{cj\omega}} \]  

(4)

With \( \omega = 2\pi f \). This impedance is similar to that of the M element of ZFit.

\[ Z_M = R_d \frac{\coth(L_d \sqrt{r_d j\omega})}{\sqrt{r_d j\omega}} \]

(5)

II - 2 OPEN-CIRCUITED URQ

Replacing c elements by q elements, with \( Z_q = \frac{1}{q(j\omega)^a} \) leads to transmission line shown in Fig. 4.

![Figure 4: L-long open uniform distributed RQ|(URQ) transmission line.](image)

The transmission line impedance is given by

\[ Z = \sqrt{r} \frac{\coth(L\sqrt{r q j\omega})^{a/2}}{\sqrt{q(j\omega)^{a/2}}} \]  

(6)

The impedance is similar to that of the M\(_a\) element of ZFit.

\[ Z_{M_a} = R \frac{\coth(\tau j\omega)^{a/2}}{(\tau j\omega)^{a/2}} \]  

(7)

With \( R = Lr, \tau = (L^2 r q)^{1/a} \)

As an example a Nyquist impedance diagram of a battery Ni-MH 1900 mAh is shown in Fig. 5. The equivalent circuit \( R1+L1+Q1/(R2+Ma3) \), containing a M\(_a\) element, is chosen to fit the data shown in Fig. 5. The values of the parameters, obtained using the ZFit tool of EC-Lab, are \( R1 = 0.049 \ \Omega, \ L1 = 0.154 \ \times 10^{-6} \ \text{H}, \ Q1 = 0.66 \ \text{F}, \ a1 = 0.61, \ R2 = 0.0236 \ \Omega, \ R3 = Lr = 0.057 \ \Omega, \ \tau3 = (L^2 r q)^{1/a} = 2.25 \ \text{s} \) and \( a3 = 0.89 \).

![Figure 5: Nyquist impedance diagram of a battery Ni-MH 1900 mAh.](image)

The equivalent circuit of the so-called anomalous diffusion is shown in Fig. 6 [6].

![Figure 6: L-long open uniform distributed QC (UQC) transmission line. Anomalous diffusion [6].](image)

The anomalous diffusion impedance is given by
\[ X = \frac{1}{q(j\omega)}^{\gamma} \zeta = \frac{1}{c j\omega} \]

\[
\coth \left( L \sqrt{\frac{c}{q}} \frac{1}{2} j\omega \right) \frac{1}{2} \]

\[ Z = \frac{\sqrt{cz}}{\sqrt{c q}} \left( \frac{1}{2} j\omega \right) \]

This impedance is similar to that of the M_{\theta} element of ZFit

\[ Z_{M_{\theta}} = R \frac{\coth \left( \frac{1}{2} \frac{c}{j\omega} \right)}{\left( \frac{1}{2} \frac{c}{j\omega} \right)} \]

(9)

With \( \gamma = 1 - \alpha \), \( R = c^\gamma \frac{L'}{q} \), \( \tau = c^\gamma \frac{L'}{q} \).

### III – SHORT-CIRCUITED TRANSMISSION LINES \( Z_l = 0 \)

#### III - 1 SHORT-CIRCUITED URC

Figure 7: L-long open uniform distributed QC (UQC) transmission line. Anomalous diffusion [6].

Using Eq. (2), the impedance of the short-circuited transmission line made of \( r \) and \( c \) elements (Fig. 7) is given by

\[ X = r, \zeta = \frac{1}{c j\omega} \]

\[ Z = \frac{\theta \left( L \sqrt{c_{j\omega}} \right)}{R \sqrt{c_{j\omega}}} \]

(10)

This impedance is similar to that of the \( W_d \) element of ZFit

\[ Z_{W_d} = R \frac{\theta \left( \frac{1}{2} \frac{c_{j\omega}}{j\omega} \right)}{\sqrt{c_{j\omega}}} \]

\( R_d = L r, \tau_d = L' r c \)

(11)

### IV – SEMI-INFINITE TRANSMISSION LINES: \( L \rightarrow \infty \)

#### IV - 1 SEMI-INFINITE URC

The impedance of the semi-infinite transmission line shown in Fig. 1 is obtained making \( L \rightarrow \infty \) in Eq. (10).

\[ L \rightarrow \infty \Rightarrow Z = \frac{\theta \left( L \sqrt{c_{j\omega}} \right)}{R \sqrt{c_{j\omega}}} \approx \frac{\sqrt{r}}{\sqrt{c_{j\omega}}} \]

(12)

This expression is similar to that of the Warburg (W) element of ZFit

\[ Z_w = \frac{2\sigma}{\sqrt{j\omega}} \text{ with } \sigma = \frac{\sqrt{r}}{2c} \]

(13)

As an example a Nyquist impedance diagram of a Fe(II)/Fe(III) system is shown in Fig. 8.

Figure 8: Nyquist impedance diagram of a Fe(II)/Fe(III) system in basic medium.

The Randles circuit \( R_1 + Q_2 / (R_2 + W_2) \), containing a Warburg element, is chosen to fit the data shown in Fig. 8. The values of the parameters for equivalent circuit are \( R_1 = 47.57 \Omega, Q_2 = 17.09 \times 10^{-6} \text{ F s}^{-1}, \alpha = 0.885, R_2 = 70.94 \Omega \) and \( \sigma_2 = 85.33 \Omega \text{ s}^{-1/2} \)

\[ \Rightarrow \sqrt{c} = 42.7 \Omega \text{ s}^{-1/2} \]

#### IV - 2 SEMI-INFINITE URRC

First of all, let us calculate the impedance of the L-long URRC transmission line (Fig. 9)
corresponding to diffusion-reaction and diffusion-trapping impedance [7]:

\[ \chi = r_1, \zeta = \frac{r_2}{1 + r_2c(j\omega)} \]

\[ \Rightarrow Z = \sqrt{r_1r_2} \left( \frac{L}{r_2} \left( 1 + r_2c(j\omega) \right) \right) \left( \frac{1}{\sqrt{1 + r_2c(j\omega)}} \right) \]

(14)

Replacing c elements by q elements

\[ \chi = r_1, \zeta = \frac{r_2}{1 + r_2c(j\omega)^{\alpha}} \]

\[ \Rightarrow Z = \sqrt{r_1r_2} \frac{th \left( L \sqrt{r_1r_2} \left( 1 + r_2c(j\omega)^{\alpha} \right) \right)}{\sqrt{1 + r_2c(j\omega)^{\alpha}}} \]

(17)

This expression is similar to that of the G\text{a} element of ZFit

\[ Z_{G\text{a}} = \frac{R}{\sqrt{1 + \tau(j\omega)^{\alpha}}}, R = \sqrt{r_1r_2}, \tau = r_2q \]

(19)

V – CONCLUSION

Seven elements, W, Wd, M, Ma, Mg, G and Ga, available in ZFit correspond to different transmission lines (Tabs. I and II).

Table I: Summary table.

<table>
<thead>
<tr>
<th>Transmission line</th>
<th>ZFit Element</th>
</tr>
</thead>
<tbody>
<tr>
<td>Open Circuited</td>
<td>URC</td>
</tr>
<tr>
<td>URC</td>
<td>M</td>
</tr>
<tr>
<td>URCQ</td>
<td>Ma</td>
</tr>
<tr>
<td>UQC</td>
<td>Mg</td>
</tr>
<tr>
<td>Short circuited</td>
<td>URC</td>
</tr>
<tr>
<td>URC</td>
<td>W</td>
</tr>
<tr>
<td>URCQ</td>
<td>G</td>
</tr>
<tr>
<td>URRQ</td>
<td>Ga</td>
</tr>
<tr>
<td>Semi-infinite</td>
<td>URC</td>
</tr>
<tr>
<td>URCQ</td>
<td>W</td>
</tr>
<tr>
<td>URRQ</td>
<td>G</td>
</tr>
<tr>
<td>Data files can be found in:</td>
<td></td>
</tr>
<tr>
<td>C: \Users\xxx\Documents\EC-Lab\Data\Samples\EIS\PEIS_Fe_Basique_1</td>
<td></td>
</tr>
</tbody>
</table>
and

<table>
<thead>
<tr>
<th>ZFit Element</th>
<th>Equations</th>
<th>Transmission Line</th>
</tr>
</thead>
</table>
| M            | $R_a = \frac{\text{coth}(\sqrt{\tau_a}j\omega)}{\sqrt{\tau_a}j\omega}$ | ![Transmission Line Diagram](image1)
|              | $R_a = L \tau, \tau_a = L^2 \tau c$ |               |
| $M_a$        | $R = \frac{\text{coth}(\tau j\omega)\alpha/2}{(\tau j\omega)^{\alpha/2}} \frac{R = L \tau}{\tau = (L^2 \tau q)^{1/\alpha}}$ | ![Transmission Line Diagram](image2)
| $M_g$        | $R = \frac{\text{coth}(\tau j\omega)\gamma/2}{(\tau j\omega)^{\gamma/2}} \frac{R = L^2 \gamma^{-1}q^{-1}/\gamma}{\tau = \gamma^{-1}L^2 q^{-1}/\gamma}$ | ![Transmission Line Diagram](image3)
| $W_d$        | $R_a = \frac{\text{th}(\sqrt{\tau_a}j\omega)}{\sqrt{\tau_a}j\omega}$ | ![Transmission Line Diagram](image4)
|              | $R_a = L \tau, \tau_a = L^2 \tau c$ |               |
| W            | $\sigma = \frac{2\sigma}{\sqrt{3} \omega}$ | ![Transmission Line Diagram](image5)
| G            | $R_G = \frac{\sqrt{1 + \tau_G j\omega}}{\sqrt{1 + \tau_G j\omega}} \frac{R_G = \sqrt{\tau_1 \tau_2}}{\tau_G = \tau_2 c}$ | ![Transmission Line Diagram](image6)
| $G_B$        | $R_G = \frac{\sqrt{1 + G_B (j\omega)^2}}{\sqrt{1 + G_B (j\omega)^2}} \frac{R_G = \sqrt{\tau_1 \tau_2}}{\tau_G = \tau_2 q}$ | ![Transmission Line Diagram](image7)

Table II: ZFit elements vs. transmission lines

References


Revised in 11/2018