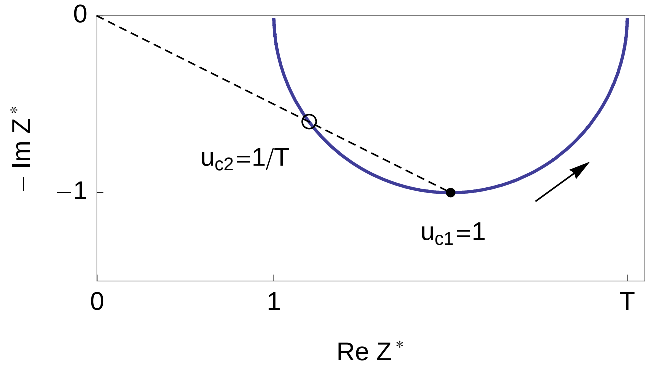
Handbook
of
Electrochemical Impedance Spectroscopy



CIRCUITS
made of
RESISTORS and INDUCTORS

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Chapter 1

Circuits containing one L

1.1 Circuit (R+L)

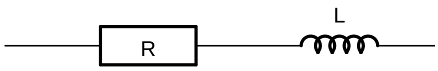

Figure 1.1: Circuit (R+L).

1.1.1 Impedance

$$Z(\omega) = R + \mathrm{i}\, L\, \omega \ , \ \mathrm{Re}\ Z(\omega) = R \ , \ \mathrm{Im}\ Z(\omega) = L\, \omega$$

1.1.2 Reduced impedance

$$Z^*(u) = Z/R = 1 + \mathrm{i}\, u \ , \ u = \omega\, \tau \ , \ \tau = L/R$$

$$\mathrm{Re}\ Z^*(u) = 1 \ , \ \mathrm{Im}\ Z^*(u) = u$$

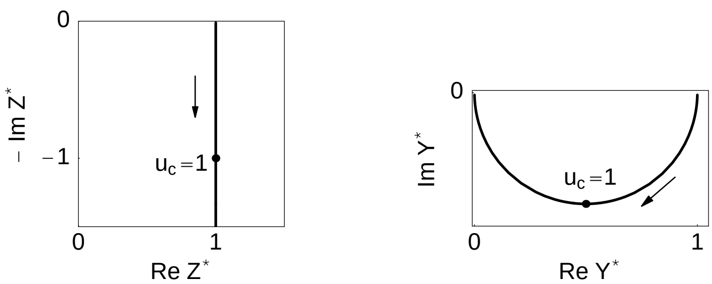

Figure 1.2: Nyquist diagrams of reduced impedance and admittance ($Y^* = R\,Y$) for the (R+L) circuit.

1.2 Circuit (R/L)

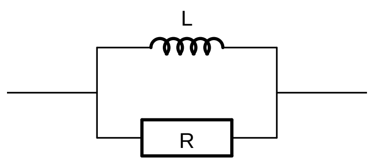

Figure 1.3: Circuit (R/L).

1.2.1 Impedance

$$Z(\omega) = \frac{1}{\dfrac{1}{\mathrm{i}\,\omega\,L} + \dfrac{1}{R}} = \frac{\mathrm{i}\,L\,R\,\omega}{R + \mathrm{i}\,L\,\omega}$$

$$\mathrm{Re}\ Z(\omega) = \frac{L^2\,R\,\omega^2}{R^2 + L^2\,\omega^2}\ ,\ \mathrm{Im}\ Z(\omega) = \frac{L\,R^2\,\omega}{R^2 + L^2\,\omega^2}$$

1.2.2 Reduced impedance

$$Z^*(u) = \frac{Z}{R} = \frac{\mathrm{i}\,u}{1 + \mathrm{i}\,u}\ ,\ u = \omega\,\tau\ ,\ \tau = L/R$$

$$\mathrm{Re}\ Z^*(u) = \frac{u^2}{1 + u^2}\ ,\ \mathrm{Im}\ Z^*(u) = \frac{u}{1 + u^2}$$

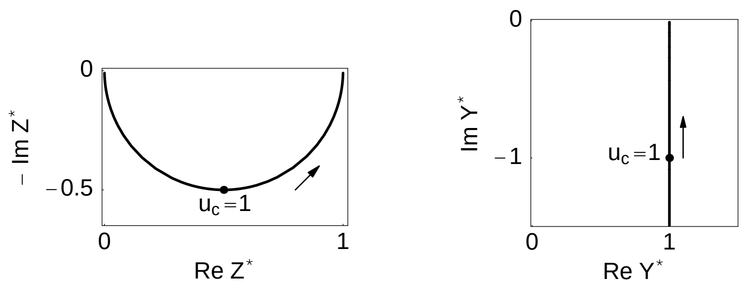

Figure 1.4: Nyquist diagrams of reduced impedance and admittance ($Y^* = R\,Y$) for the (R/L) circuit.

1.3 Circuit (R2+(R1/L1))

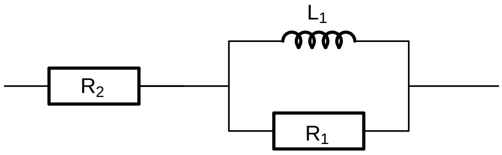

Figure 1.5: Circuit (R2+(R1/L1)).

1.3.1 Impedance

$$Z(\omega) = R_2 + \frac{1}{\frac{1}{\mathrm{i}\,\omega\,L_1} + \frac{1}{R_1}} = \frac{R_2\left(1 + \frac{\mathrm{i}\,\omega\,L_1\,(R_1 + R_2)}{R_1\,R_2}\right)}{1 + \frac{\mathrm{i}\,\omega\,L_1}{R_1}}$$

$$\mathrm{Re}\;Z(\omega) = \frac{\omega^2\,{L_1}^2\,R_1}{\omega^2\,{L_1}^2 + {R_1}^2} + R_2\;,\;\mathrm{Im}\;Z(\omega) = \frac{\omega\,L_1\,{R_1}^2}{\omega^2\,{L_1}^2 + {R_1}^2}$$

1.3.2 Reduced impedance

$$Z^*(u) = Z/R_2 = \frac{1 + \mathrm{i}\,T\,u}{1 + \mathrm{i}\,u}\;,\;u = \tau_1\,\omega\;,\;\tau_1 = L_1/R_1 \tag{1.1}$$

$$T = (R_1 + R_2)/R_2 = 1 + R_1/R_2 > 1$$

$$\mathrm{Re}\;Z^*(u) = \frac{1 + T\,u^2}{1 + u^2}\;,\;\mathrm{Im}\;Z^*(u) = \frac{(-1 + T)\;u}{1 + u^2}$$

$$\lim_{u \to 0} \mathrm{Re}\;Z^*(u) = 1\;,\;\lim_{u \to \infty} \mathrm{Re}\;Z^*(u) = T$$

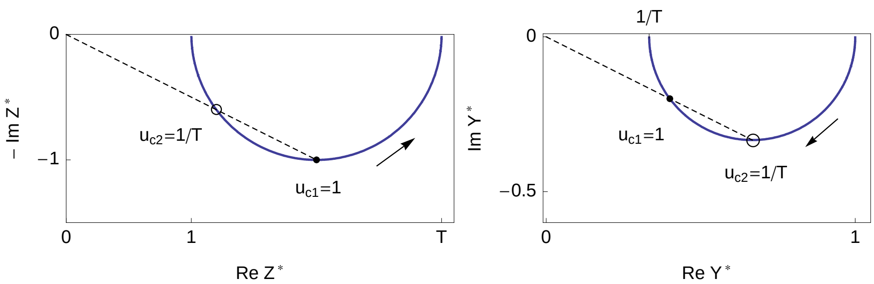

Figure 1.6: Nyquist diagram of reduced impedance and admittance ($Y^* = R_2\,Y$) for the (R2+(R1/L1)) circuit, plotted for $T = 3$.

1.4 Circuit $((R_1+L_1)/R_2)$

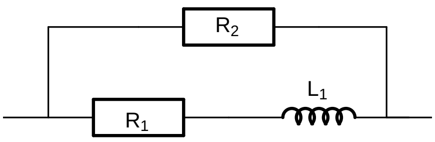

Figure 1.7: Circuit $((R_1+L_1)/R_2)$.

1.4.1 Impedance

$$Z(\omega) = \frac{(\mathrm{i}\,\omega\,L_1 + R_1)\;R_2}{\mathrm{i}\,\omega\,L_1 + R_1 + R_2} = \frac{R_1\,R_2\left(1+\dfrac{\mathrm{i}\,\omega\,L_1}{R_1}\right)}{(R_1+R_2)\left(1+\dfrac{\mathrm{i}\,\omega\,L_1}{R_1+R_2}\right)}$$

$$\mathrm{Re}\;Z(\omega) = \frac{R_2\,\left(\omega^2\,{L_1}^2 + R_1\;(R_1+R_2)\right)}{\omega^2\,{L_1}^2 + (R_1+R_2)^2}\;,\;\mathrm{Im}\;Z(\omega) = \frac{\omega\,L_1\,{R_2}^2}{\omega^2\,{L_1}^2 + (R_1+R_2)^2}$$

1.4.2 Reduced impedance

$$Z^*(u) = \frac{Z\;(R_1+R_2)}{R_1\,R_2} = \frac{1+\mathrm{i}\,T\,u}{1+\mathrm{i}\,u}\;,\;u=\tau_1\,\omega\;,\;\tau_1 = L_1/(R_1+R_2)$$

$$T=\tau_2/\tau_1\;,\;\tau_2 = L_1/R_1\;,\;T = 1+R_2/R_1 > 1$$

cf. § 1.3.2, Eq. (1.1) and Fig. 1.6.

1.5 Transformation formulae (R+(R/L)) ↔ ((R+L)/R)

Figure 1.8: The (R+(R/L)) and ((R+L)/R) circuits are non-distinguishable.

1.5.1 Transformation formulae (R+(R/L)) → ((R+L)/R)

$$R_{22} = R_{11} + R_{21}, L_{12} = \frac{L_{11}}{R_{11}}, R_{12} = R_{21} + \frac{{R_{21}}^2}{R_{11}}$$

1.5.2 Transformation formulae ((R+L)/R) → (R+(R/L))

$$L_{11} = \frac{L_{12}\,{R_{22}}^2}{R_{12} + R_{22}}, R_{11} = \frac{{R_{22}}^2}{R_{12} + R_{22}}, R_{21} = \frac{R_{12}\,R_{22}}{R_{12} + R_{22}}$$

1.6 Circuits containing L vs. circuits containing C

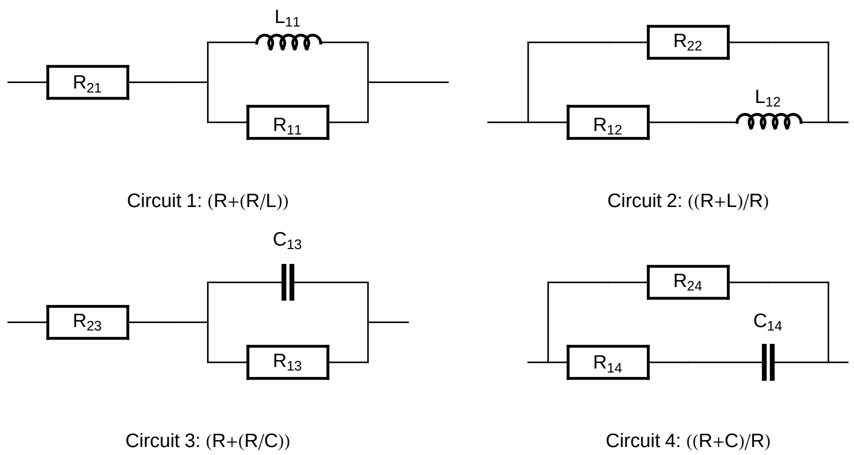

Figure 1.9: (R+(R/L)), ((R+L)/R), (R+(R/C)), ((R+C)/R) circuits are non-distinguishable, taking account of negative values of parameters.

1.6.1 Transformation formulae circuit 1 ↔ circuit 2 and circuit 3 ↔ circuit 4

cf. § 1.5 and "Handbook of EIS : Circuits made of Rs and Cs".

1.6.2 Transformation formulae circuit 3 → circuit 1

$$L_{11} = -C_{13}\,{R_{13}}^2, R_{21} = R_{13} + R_{23}, R_{11} = -R_{13}$$

1.6.3 Transformation formulae circuit 1 → circuit 3

$$C_{13} = -\frac{L_{11}}{{R_{11}}^2}, R_{23} = R_{11} + R_{21}, R_{13} = -R_{11}$$

1.6.4 Transformation formulae circuit 4 → circuit 1

$$L_{11} = -C_{14}\,{R_{24}}^2,\; R_{21} = R_{24}, R_{11} = -\frac{{R_{24}}^2}{R_{14} + R_{24}}$$

1.6.5 Transformation formulae circuit 1 → circuit 4

$$R_{24} = R_{21}, R_{14} = -\frac{R_{21}\ (R_{11} + R_{21})}{R_{11}}, C_{14} = -\frac{L_{11}}{{R_{21}}^2}$$

1.6.6 Transformation formulae circuit 3 → circuit 2

$$L_{12} = C_{13}\, R_{13}, R_{22} = R_{23}, R_{12} = -\frac{R_{23}\ (R_{13} + R_{23})}{R_{13}}$$

1.6.7 Transformation formulae circuit 2 → circuit 3

$$C_{13} = -\frac{L_{12}\ (R_{12} + R_{22})}{{R_{22}}^2}, R_{23} = R_{22},\ R_{13} = -\frac{{R_{22}}^2}{R_{12} + R_{22}}$$

1.6.8 Transformation formulae circuit 4 → circuit 2

$$L_{12} = C_{14}\ (R_{14} + R_{24}), R_{22} = \frac{R_{14}\, R_{24}}{R_{14} +\ R_{24}}, R_{12} = -R_{14}$$

1.6.9 Transformation formulae circuit 2 → circuit 4

$$R_{14} = -R_{12}, R_{24} = \frac{R_{12}\ \ R_{22}}{R_{12} + R_{22}}, C_{14} = -\frac{L_{12}\ (R_{12} + R_{22})}{{R_{12}}^2}$$

1.7 Modified inductance

$$Z = L_\alpha\, (\mathrm{i}\, \omega)^\alpha\ ,\ \mathrm{Re}\ Z = L_\alpha\, \omega^\alpha\, c_\alpha\ ,\ \mathrm{Im}\ Z = L_\alpha\, \omega^\alpha\, s_\alpha$$

$$c_\alpha = \cos(\frac{\pi\,\alpha}{2})\ ,\ s_\alpha = \sin(\frac{\pi\,\alpha}{2})$$

$$|Z| = L_\alpha\, \omega^\alpha\ ,\ \phi_Z = \frac{\pi\,\alpha}{2} = \mathrm{cte}$$

The L_α unit ($\mathrm{H\ cm^2\, s^{1-\alpha}}$) depends on α.

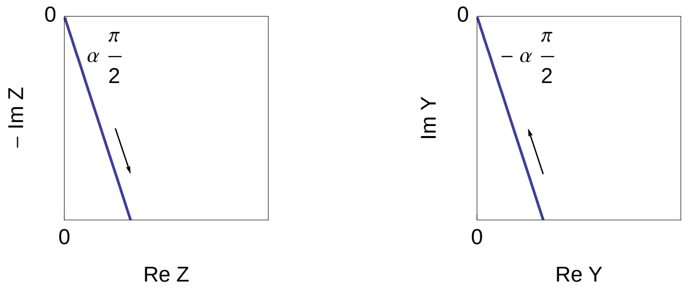

Figure 1.10: Nyquist diagrams of the impedance and admittance for the modified inductance L_α, plotted for $\alpha = 0.8$.

Chapter 2

Circuits made of one R and two Ls

2.1 Circuit $((R_1/L_1)+L_2)$

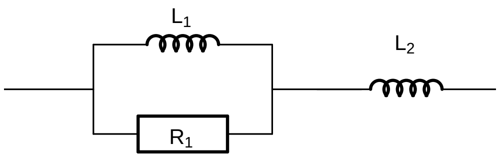

Figure 2.1: Circuit $((R_1/L_1)+L_2)$.

2.1.1 Impedance

$$Z(\omega) = \frac{1}{\frac{1}{R_1} + \frac{1}{\mathrm{i}\,\omega\, L_1}} + \mathrm{i}\,\omega\, L_2 = \frac{\mathrm{i}\,\omega\ (L_1 + L_2)\ \left(1 + \frac{\mathrm{i}\,\omega\, L_1\, L_2}{(L_1 + L_2)\ R_1}\right)}{1 + \frac{\mathrm{i}\,\omega\, L_1}{R_1}}$$

2.1.2 Time constants

$$Z(\omega) = \frac{(L_1 + L_2)\ \mathrm{i}\,\omega\ (1 + \mathrm{i}\,\omega\, \tau_2)}{1 + \mathrm{i}\,\omega\, \tau_1}\ ,\ \tau_1 = \frac{L_1}{R_1}\ ,\ \tau_2 = \frac{L_1\, L_2}{(L_1 + L_2)\, R_1}$$

$$\mathrm{Re}\ Z(\omega) = \frac{\omega^2\ (L_1 + L_2)\ (\tau_1 - \tau_2)}{1 + \omega^2\, {\tau_1}^2}\ ,\ \mathrm{Im}\ Z(\omega) = \frac{\omega\ (L_1 + L_2)\ \left(1 + \omega^2\, \tau_1\, \tau_2\right)}{1 + \omega^2\, {\tau_1}^2}$$

$$\lim_{\omega \to \infty} \mathrm{Re}\ Z(\omega) = \frac{(L_1 + L_2)\ (\tau_1 - \tau_2)}{{\tau_1}^2} = R_1$$

2.1.3 Reduced impedance

$$Z^*(u) = Z(\omega)/R_1 = \frac{1}{1-T}\frac{\mathrm{i}\,u\,(1+\mathrm{i}\,T\,u)}{1+\mathrm{i}\,u} \tag{2.1}$$

$$u = \omega\,\tau_1\ ,\ T = \frac{\tau_2}{\tau_1} = \frac{L_2}{L_1+L_2} < 1$$

$$\mathrm{Re}\ Z^*(u) = \frac{u^2}{1+u^2}\ ,\ \mathrm{Im}\ Z^*(u) = -\frac{u\,\left(1+T\,u^2\right)}{(-1+T)\,(1+u^2)}\ ,\ \lim_{u\to\infty} \mathrm{Re}\ Z(u) = 1$$

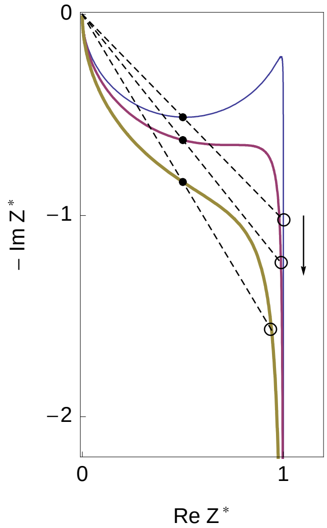

Figure 2.2: Nyquit diagrams of reduced impedance for the ((R₁/L₁)+L₂) circuit (Eq. (2.1), Fig. 2.1, $T = 1/4, 1/9, 1/90$, the line thickness inceases with increasing T). Horizontal tangent for $T \le 1/9$ ($L_1/L_2 \ge 8$). Dots: reduced caracteristic angular frequency: $u_{c1} = 1$; circles: reduced caracteristic angular frequency $u_{c2} = 1/T$.

2.2 Circuit ((R_1+L_2)/L_1)

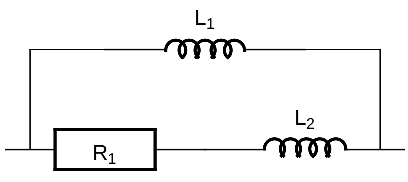

Figure 2.3: Circuit ((R_1+L_2)/L_1).

2.2.1 Impedance

$$Z(\omega) = \frac{\mathrm{i}\,\omega\,L_1\,\left(\mathrm{i}\,\omega\,L_2 + R_1\right)}{\mathrm{i}\,\omega\,L_1 + \mathrm{i}\,\omega\,L_2 + R_1} = \frac{\mathrm{i}\,\omega\,L_1\,\left(1 + \dfrac{\mathrm{i}\,\omega\,L_2}{R_1}\right)}{1 + \dfrac{\mathrm{i}\,\omega\,\left(L_1 + L_2\right)}{R_1}}$$

2.2.2 Time constants

$$Z(\omega) = \frac{\mathrm{i}\,\omega\,L_1\,\left(1 + \mathrm{i}\,\omega\,\tau_2\right)}{1 + \mathrm{i}\,\omega\,\tau_1}\ ,\ \tau_1 = \frac{L_1 + L_2}{R_1}\ ,\ \tau_2 = \frac{L_2}{R_1}$$

$$\mathrm{Re}\,Z(\omega) = \frac{\omega^2\,L_1\,\left(\tau_1 - \tau_2\right)}{1 + \omega^2\,{\tau_1}^2}\ ,\ \mathrm{Im}\,Z(\omega) = \frac{\omega\,L_1\,\left(1 + \omega^2\,\tau_1\,\tau_2\right)}{1 + \omega^2\,{\tau_1}^2}$$

$$\lim_{\omega\to\infty} \mathrm{Re}\,Z(\omega) = \frac{L_1\,\left(\tau_1 - \tau_2\right)}{{\tau_1}^2} = \frac{{L_1}^2\,R_1}{\left(L_1 + L_2\right)^2}$$

2.2.3 Reduced impedance

$$Z^*(u) = Z(\omega)/R_1 = (1 - T)\,\frac{\mathrm{i}\,u\,\left(1 + \mathrm{i}\,T\,u\right)}{1 + \mathrm{i}\,u} \tag{2.2}$$

$$u = \omega\,\tau_1\ ,\ T = \frac{\tau_2}{\tau_1} = \frac{L_2}{L_1 + L_2} < 1$$

$$\mathrm{Re}\,Z^*(u) = \frac{\left(-1 + T\right)^2\,u^2}{1 + u^2}\ ,\ \mathrm{Im}\,Z^*(u) = -\frac{\left(-1 + T\right)\,u\,\left(1 + T\,u^2\right)}{1 + u^2}$$

$$\lim_{u\to\infty} \mathrm{Re}\,Z(u) = \frac{{L_1}^2}{\left(L_1 + L_2\right)^2}$$

(Fig. 2.4)

2.3 Transformation formulae ((R/L)+L) ↔ ((R+L)/L)

2.3.1 Transformation formulae ((R/L)+L) → ((R+L)/L)

$$R_{12} = \frac{\left(L_{11} + L_{21}\right)^2\,R_{11}}{{L_{11}}^2}, L_{22} = L_{21} + \frac{{L_{21}}^2}{L_{11}}, L_{12} = L_{11} + L_{21}$$

2.3.2 Transformation formulae ((R+L)/L) → ((R/L)+L)

$$L_{21} = \frac{L_{12}\,L_{22}}{L_{12} + L_{22}}, L_{11} = \frac{{L_{12}}^2}{L_{12} + L_{22}}, R_{11} = \frac{{L_{12}}^2\,R_{12}}{\left(L_{12} + L_{22}\right)^2}$$

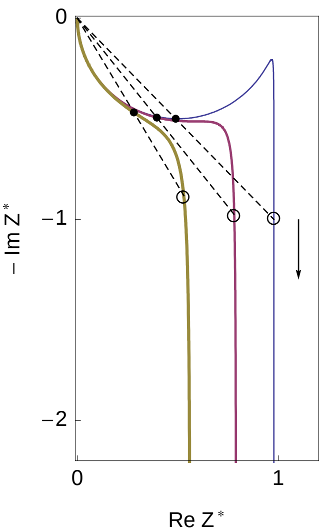

Figure 2.4: Nyquit diagrams of the reduced impedance for the ((R$_1$+L$_2$)/L$_1$) circuit (Eq. (2.2), Fig. 2.3, $T = 1/4, 1/9, 1/90$, the line thickness inceases with increasing T). Horizontal tangent for $T \leq 1/9$ ($L_1/L_2 \geq 8$). Dots: reduced caracteristic angular frequency: $u_{c1} = 1$; circles: reduced caracteristic angular frequency $u_{c2} = 1/T$.

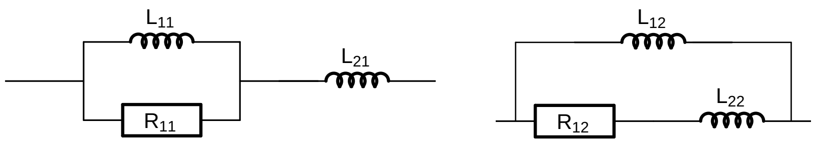

Figure 2.5: The ((R/L)+L) and ((R+L)/L) circuits are non-distinguishable.